Problem Set for the NPTEL Course
COMBINATORICS

Module 1: Pigeon Hole Principle

1. (a) Show that if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$ then there are always two which differ by 1.

   (b) Show that if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 3n\}$ then there are always two which differ by at most 2.

   (c) Generalise the above 2 statements.

2. Use the pigeon hole principle to prove that the decimal expansion of a rational number m/n eventually is repeating.

3. Prove that any 5 points chosen within a square of side length 2, there are 2 whose distance apart is at most $\sqrt{2}$.

4. There are 100 people at a party. Each person has an even number (possibly 0) of aquaintances. Prove that there are 3 people at the party with the same number of aquaintances.

5. Prove that in a graph of n vertices, where $n \geq 6$, there exists either a triangle (i.e. a complete subgraph on 3 vertices) or an independent set on 3 vertices.

Module 2: Elementary Concepts and Basic Counting Principles

1. Prove that the number of permutations of m A's and at most n B's equals

$$\binom{m+n+1}{m+1}$$

2. Consider the multiset $\{n.a, 1, 2, \ldots, n\}$ of size $2n$. Determine the number of its n-combinations.

3. Consider the multiset $\{n.a, n.b, 1, 2, \ldots, n+1\}$ of size $3n+1$. Determine the number of n-combinations.

4. Establish a bijection between the permutations of the set $\{1, 2, \ldots, n\}$ and the towers of the form $A_0 \subset A_1 \subset A_n$, where $|A_k| = k$ for $k = 0, 1, \ldots, n$.

5. A city has n junctions. It is decided that some of them will get traffic lights, and some of those that get traffic lights will also get a gas station. If at least one gas station comes up, then in howmany different ways can this happen ?

Module 3: More Strategies

1. Find the number of integers between 1 and 10,000 which are neither pefect squares nor perfect cubes.

2. Determine the number of 12-combinations of the multi-set $S = \{4.a, 3.b, 4.c, 5.d\}$.

3. Determine a general formula for the number of permutations of the set $\{1, 2, \ldots, n\}$ in which exactly k integers are in their natural positions.

4. Use a combinatorial reasoning to derive the identity:
   $n! = \sum_{i=0}^{n} \binom{n}{i} D_{n-i}$, where D_i is the number of permutations of $\{1, 2, \ldots, i\}$ such that no number is in its natural position. (D_0 is defined to be 1.)

5. Find the number of permutations of $a, b, c \ldots, x, y, z$ in which none of the patterns *spin, game, path,* and *net* occurs.

Module 4: Recurrence Relations and Generating Functions

1. Prove that the Fibonacci sequence is the solution of the recurrence relation

   $$a_n = 5a_{n-4} + 3a_{n-5}, n \geq 5$$

   where $a_0 = 0, a_1 = 1, a_2 = 1, a_3 = 2$ and $a_4 = 3$. Then use this formula to show that the Fibonacci numbers satisfy the condition that f_n (the n-th Fibonacci number $= a_n$) is divisible by 5 if and only if n is divisible by 5.

2. Consider a 1-by-n chess board. Suppose we color each square of the chess board with one of 3 colors, red, green and blue so that no two squares that are colored red are adjacent. Let h_n be the number of such colorings possible. Find and verify a recurrence relation that h_n satisfies. Then find a formula for h_n.

3. Solve the recurrence relation $h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3}$, $n \geq 3$ with initial values $h_0 = 0, h_1 = 1, h_2 = 2$.

4. Solve the non-homogenious recurrence relation $h_n = 4h_{n-1} + 3.2^n$, $n \geq 1$ with initial value, $h_0 = 1$.

5. Let h_n be the number of ways to color the squares of a 1-by-n chess board with the colors red, white, blue, and green in such a way that the number of squares colored red is even and the number of squares colored white is odd. Determine the exponential generating function for the sequence $h_0, h_1, h_2, \ldots$, and then find a simple formula for h_n.

Module 5: Special Numbers

1. Let $2n$ (equally spaced) points on a circle be chosen. Show that the number of ways to join these points in pairs, so that the resulting n line segments do not intersect, equals the nth Catalan number.

2. Consider the Sterling Number of the second kind, $S(n, k)$. Show that $S(n, n-2) = \binom{n}{3} + 3.\binom{n}{4}$.

3. The number of partitions of a set of n elements into k distinguishable boxes (some of which may be empty) is k^n. By counting in a different way, prove that

$$k^n = \sum_{i=1}^{n} \binom{k}{i} i! S(n, i)$$

   If $k > n$, define $S(n, k) = 0$.

4. For each integer $n > 2$, determine a self-conjugate partition of n that has at least two parts.

5. Consider Sterling number of the first kind, $s(n, k)$. Show that $\sum_{i=0}^{n} s(n, i) = n!$

6. Let $t_1, t_2, \ldots, t_m$ be distinct positive integers, and let $q_n = q_n(t_1, t_2, \ldots, t_n)$ equal the number of partitions of n in which all parts are taken from $t_1, t_2, \ldots, t_m$. Define $q_0 = 1$. Show that the generating function for the sequence $q_0, q_1, q_2, \ldots,$ is $\prod_{k=1}^{m}(1 - x^{t_k})^{-1}$.